数理解析研究所講究録 1414 巻 2005 年 40-45

A note on the length of starlike functions

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(Memorial Paper for Professor Nicolae N. Pascu)

Abstract

Let $\mathcal{S}$ be the class of analytic functions $f(z)$ normalized with $f(0) = 0$ and $f'(0) = 1$ which are univalent in the open unit disk $\mathbb{U}$. Also, let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of functions $f(z)$ which are starlike with respect to the origin in $\mathbb{U}$. For $f(z) \in \mathcal{S}^*$, Ch. Pommerenke [J. London Math. Soc. 37(1962), 209 - 224] has shown the estimates for the length of the image curve of the circle $|z| = r < 1$. The object of the present paper is to derive the generalized theorem of the result due to Ch. Pommerenke.

1 Introduction

Let $\mathcal{S}$ denote the set of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} | \quad |z| < 1\}$. A function $f(z) \in \mathcal{S}$ is called starlike with respect to the origin if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of all starlike functions with respect to the origin in $\mathbb{U}$. In 1962, Pommerenke [6] has shown

Theorem A *Let $f(z) \in \mathcal{S}^*$ and suppose that*

$$M(r) = \mathrm{Max}_{|z|=r<1} |f(z)| \leqq \frac{1}{(1-r)^{\alpha}} \qquad (0 < \alpha \leqq 2).$$

Then

$$L(r) = \int_0^{2\pi} r\left|f'(re^{i\theta})\right| d\theta \leqq \frac{A(\alpha)}{(1-r)^{\alpha}},$$

where $A(\alpha)$ depends only on α and $L(r)$ denotes the length of $C(r)$ which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$.

2004 *Mathematics Subject Classifications*: Primary 30C45.
Key Words and Phrases: Univalent function, starlike function, length, subordination.

Pommerenke [6, Remarks in p.214] has given the comments that Theorem A can not be improved any more, except for the factor $A(\alpha)$. This is true, but it is not absolutely perfect, because the order of infinity for $M(r)$ depends not only $(1-r)^{-\alpha}$ but $(\log(1-r)^{-1})^{\beta}$.

In 1958, Hayman [1] has proved that if $f(z) \in \mathcal{S}$ and $1/2 < \alpha \leqq 2$, then

$$M(r) = O\left(\left(\frac{1}{1-r}\right)^{\alpha}\right)$$

implies that

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^{\alpha}\right).$$

Littlewood [4] has shown that this implication breaks down for small α. On the other hand, Thomas [7] has obtained

Theorem B *Let $f(z) \in \mathcal{S}^*$. Then*

$$L(r) = O\left(\sqrt{B(r)}\log\left(\frac{1}{1-r}\right)\right) \qquad (as \quad r \to 1),$$

where $B(r)$ is the area enclosed by the curve $C(r)$ which is the image curve of the circle $|z| = r < 1$ under the mapping $w = f(z)$.

It is the purpose of this paper to generalize Theorem A by Pommerenke [6].

2 Main theorem

To discuss our main theorem, we need the following lemma due to Pommerenke [6] (or also due to Hayman [1]).

Lemma *If $f(z) \in \mathcal{S}$, then, for $\lambda > 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\lambda} dt \leqq \lambda^2 \int_0^r \frac{M(\rho)^{\lambda}}{\rho}\, d\rho \qquad (0 < r < 1).$$

Now, we give

Theorem *Let $f(z) \in \mathcal{S}^*$ and suppose that*

$$M(r) = \mathrm{Max}_{|z|=r<1}\,|f(z)| = O\left(\left(\frac{1}{1-r}\right)^{\alpha}\left(\log\frac{1}{1-r}\right)^{\beta}\right),$$

where $0 < \alpha < k \leqq 2$, $k > 1$, and $\beta > 0$. Then we have

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^{\alpha}\left(\log\frac{1}{1-r}\right)^{\beta+1-\frac{\alpha}{k}}\right)$$

for $0 < \alpha < k-1$, *and*

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^{2\alpha\left(1-\frac{1}{k}\right)+2-k}\left(\log\frac{1}{1-r}\right)^{\beta}\right)$$

for $0 < k-1 \leqq \alpha < k$.

Proof Application of the Hölder's inequality gives us that

$$\begin{aligned} L(r) &= \int_0^{2\pi} r\left|f'(re^{i\theta})\right| d\theta = \int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right||f(z)|\,d\theta \\ &\leqq \left(\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^{\frac{k}{k-\alpha}} d\theta\right)^{\frac{k-\alpha}{k}}\left(\int_0^{2\pi}|f(z)|^{\frac{k}{\alpha}}\,d\theta\right)^{\frac{\alpha}{k}} \\ &= I^{\frac{k-\alpha}{k}} J^{\frac{\alpha}{k}}, \end{aligned}$$

where $k > \alpha$,

$$I = \int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^{\frac{k}{k-\alpha}} d\theta$$

and

$$J = \int_0^{2\pi}|f(z)|^{\frac{k}{\alpha}}\,d\theta.$$

By Keogh [2, Theorem 1], it is well known that

$$\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right| d\theta = O\left(\log\frac{1}{1-r}\right) \qquad (\text{as} \quad r \to 1).$$

On the other hand, we see that

$$\left|\frac{zf'(z)}{f(z)}\right| = O\left(\frac{1}{1-r}\right) \qquad (\text{as} \quad r \to 1)$$

by Nehari [5]. Thus, we have the following estimates for $0 < \alpha < k-1$ that

$$\begin{aligned} I &= \int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|\left|\frac{zf'(z)}{f(z)}\right|^{\frac{\alpha}{k-\alpha}} d\theta \\ &= \left(O\left(\frac{1}{1-r}\right)^{\frac{\alpha}{k-\alpha}}\right)\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right| d\theta \\ &= O\left(\left(\frac{1}{1-r}\right)^{\frac{\alpha}{k-\alpha}}\left(\log\frac{1}{1-r}\right)\right) \qquad (\text{as} \quad r \to 1), \end{aligned}$$

which shows that

$$I^{\frac{k-\alpha}{k}} = O\left(\left(\frac{1}{1-r}\right)^{\frac{\alpha}{k}}\left(\log\frac{1}{1-r}\right)^{\frac{k-\alpha}{k}}\right).$$

In order to consider for the case $0 < k-1 \leqq \alpha < k$, we have to recall here the following result by Littlewood [3, p.484] that if $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$, then for each r $(0 \leqq r < 1)$ and each k $(k \geqq 0)$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^k d\theta \leqq \int_0^{2\pi} \left|F(re^{i\theta})\right|^k d\theta.$$

Since $f(z) \in \mathcal{S}^*$, we see that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \qquad (z \in \mathbb{U}),$$

where the symbol $\prec$ means the subordination. Applying the result by Littlewood [3], we have for $1 < k \leqq 2$ that

$$\int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^k d\theta \leqq \int_0^{2\pi} \left|\frac{1+z}{1-z}\right|^k d\theta \leqq \int_0^{2\pi} \left|\frac{1+z}{1-z}\right|^2 d\theta$$

$$= O\left(\frac{1}{1-r}\right) \qquad (\text{as} \quad r \to 1).$$

Therefore, for the case of $0 < k-1 \leqq \alpha < k$,

$$I = \int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^k \left|\frac{zf'(z)}{f(z)}\right|^{\frac{k(\alpha-k+1)}{k-\alpha}} d\theta$$

$$= \left(O\left(\frac{1}{1-r}\right)^{\frac{k(\alpha-k+1)}{k-\alpha}}\right) \int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^k d\theta$$

$$= O\left(\frac{1}{1-r}\right)^{\frac{(k-1)(\alpha-k+1)+1}{k-\alpha}} \qquad (\text{as} \quad r \to 1),$$

which implies that

$$I^{\frac{k-\alpha}{k}} = O\left(\frac{1}{1-r}\right)^{(1-\frac{1}{k})\alpha-(k-2)}.$$

Next, we have to consider J by using the lemma due to Pommerenke [6]. By using the lemma and Schwarz lemma, we have, for $0 < \alpha < k$, that

$$J = \int_0^{2\pi} |f(z)|^{\frac{k}{\alpha}}\, d\theta \leqq \frac{2k^2\pi}{\alpha^2} \int_0^r \frac{1}{\rho} M(\rho)^{\frac{k}{\alpha}} d\rho$$

$$\leqq \frac{2k^2\pi}{\alpha^2}\int_0^r \frac{1}{\rho}\left\{\frac{\rho}{(1-\rho)^\alpha}\left(\log\frac{1}{1-\rho}\right)^\beta\right\}^{\frac{k}{\alpha}} d\rho$$

$$= \frac{2k^2\pi}{\alpha^2}\int_0^r \frac{\rho^{\frac{k}{\alpha}-1}}{(1-\rho)^k}\left(\log\frac{1}{1-\rho}\right)^{\frac{k\beta}{\alpha}} d\rho$$

$$\leqq \frac{2k^2\pi}{\alpha^2}\int_0^r \left(\frac{1}{1-\rho}\right)^k\left(\log\frac{1}{1-\rho}\right)^{\frac{k\beta}{\alpha}} d\rho$$

$$\leqq \frac{2k^2\pi}{\alpha^2}\left(\log\frac{1}{1-r}\right)^{\frac{k\beta}{\alpha}}\int_0^r \left(\frac{1}{1-\rho}\right)^k d\rho$$

$$= O\left(\left(\frac{1}{1-r}\right)^{k-1}\left(\log\frac{1}{1-r}\right)^{\frac{k\beta}{\alpha}}\right) \qquad (\text{as } \; r \to 1),$$

which gives us that

$$J^{\frac{\alpha}{k}} = O\left(\left(\frac{1}{1-r}\right)^{\frac{\alpha(k-1)}{k}}\left(\log\frac{1}{1-r}\right)^\beta\right).$$

Consequently, we conclude that, for $0 < \alpha < k-1$,

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^\alpha\left(\log\frac{1}{1-r}\right)^{\beta+1-\frac{\alpha}{k}}\right),$$

and, for $0 < k-1 \leqq \alpha < k$,

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^{2\alpha(1-\frac{1}{k})+(2-k)}\left(\log\frac{1}{1-r}\right)^\beta\right).$$

This completes the proof of our main theorem.

Taking $k = 2$ in Theorem, we have

Corollary *Let $f(z) \in \mathcal{S}^*$ and suppose that*

$$M(r) = \mathrm{Max}_{|z|=r<1}|f(z)| = O\left(\left(\frac{1}{1-r}\right)^\alpha\left(\log\frac{1}{1-r}\right)^\beta\right),$$

where $0 < \alpha < 2$ and $\beta > 0$. Then we have

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^\alpha\left(\log\frac{1}{1-r}\right)^{\beta+1-\frac{\alpha}{2}}\right) \qquad (\text{for} \quad 0 < \alpha < 1)$$

and

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^\alpha\left(\log\frac{1}{1-r}\right)^\beta\right) \qquad (\text{for} \quad 1 \leqq \alpha < 2).$$

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